UNIVERSITY OF ALBERTA
LIBRARY

Regulations Regarding Theses and Dissertations

Typescript copies of theses and dissertations for Master's and Doctor's degrees deposited in the University of Alberta Library, as the official Copy of the Faculty of Graduate Studies, may be consulted in the Reference Reading Room only.

A second copy is on deposit in the Department under whose supervision the work was done. Some Departments are willing to loan their copy to libraries, through the inter-library loan service of the University of Alberta Library.

These theses and dissertations are to be used only with due regard to the rights of the author. Written permission of the author and of the Department must be obtained through the University of Alberta Library when extended passages are copied. When permission has been granted, acknowledgement must appear in the published work.

This thesis or dissertation has been used in accordance with the above regulations by the persons listed below. The borrowing library is obligated to secure the signature of each user.

Please sign below:

Date	Signature	Institution

GENERALIZATIONS OF SOME FIXED POINT THEOREMS IN METRIC SPACES

BY

© KAI-WANG NG

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

THE UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA

SPRING, 1969

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "GENERALIZATIONS OF SOME FIXED POINT THEOREMS IN METRIC SPACES", submitted by KAI-WANG NG in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis consists of generalizations and applications of some fixed point theorems in metric space which have appeared recently in the literature.

In Chapter I, we give a result concerning the cluster sets of non-expansive mappings and apply it to generalize a theorem of M. Edelstein.

Chapter II is devoted to a study of the convergence of fixed points of a sequence of mappings. Some applications to differential and integral equations are given. The previous results of F.F. Bonsall and S.B. Nadler are considerably generalized.

In the last Chapter we make the observation that the type of contractive mapping (mappings which shrink distance in some manner) hypotheses treated in the literature are very special in the sense that they all satisfy a rather severe condition: every periodic point must necessarily be a fixed point.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Dr. T. D. Rogers, for his guidance and direction during the preparation of this thesis.

I am also indebted to Dr. S. A. Naimpally who lent me his collection of references.

TABLE OF CONTENTS

CHAPTER I

THE CLUSTER SET OF A NON-EXPANSIVE MAPPING

Let T be a mapping from a metric space X into itself. For any $x \varepsilon X$ we denote $T^o x = x$, $T^{n+1}x = T(T^n x)$, $n = 0,1,2,\ldots$. The cluster set $\ell(x)$ of a point x in X is defined to be the set of all limits of convergent subsequences of $\{T^n x\}_{n=o}^{\infty}$.

A mapping $T:T \to X$ is called non-expansive if $d(Tx,Ty) \leq d(x,y)$ for $x,y \varepsilon X$. Here we give a result concerning the cluster set of a non-expansive mapping and apply it to modify a theorem of Edelstein [5].

<u>Theorem I.1.</u>

Let $T:X \to X$ be a non-expansive mapping. If $\ell(x)$ contains a fixed point ξ of T, then

$$\xi = \lim_{n \to \infty} T^n x \quad ,$$

and consequently $\ell(x) = \{\xi\}$. In particular, every cluster set $\ell(x)$ under a non-expansive mapping contains at most one fixed point.

<u>Proof</u>

Suppose $T\xi = \xi \varepsilon \ell(x)$. Then given $\varepsilon > 0$, there exists a positive integer N such that

$$d(T^N x,\xi) < \varepsilon .$$

Hence for $n \geq N$, we have

$$\begin{aligned} d(T^n x,\xi) &= d(T^n x,T^n \xi) \\ &\leq d(T^N x,T^N \xi) \\ &= d(T^N x,\xi) < \varepsilon , \end{aligned}$$

since T is non-expansive and $T\xi = \xi$. Therefore, $\xi = \lim_{n \to \infty} T^n x$. Since the sequence $\{T^n x\}_{n=0}^{\infty}$ converges to ξ, so does every subsequence, hence $\ell(x) = \{\xi\}$.

Theorem I.2.

Let $T:X \to X$ be a non-expansive mapping satisfying Bailey's condition [1]:

for any $x,y \in X$, $x \neq y$, there exists a positive integer $n = n(x,y)$ (depending on x,y), such that $d(T^n x,T^n y) < d(x,y)$.

Then each $\xi \in \ell(x)$ is a unique fixed point of T.

Proof.

It is clear that any mapping satisfying Bailey's condition can have at most one fixed point. Indeed, if $Tx = x$, $Ty = y$ and $x \neq y$, then there is an n such that $d(T^n x,T^n y) < d(x,y)$, a contradiction.

Let $\xi \in \ell(x)$, then since T is non-expansive, by a theorem of Edelstein [6], ξ generates an isometric sequence. This means for any integers $m > 0$, $n > 0$,

$$d(T^m\xi, T^n\xi) = d(T^{m+k}\xi, T^{n+k}\xi) \ , \qquad k = 1,2,\ldots \ .$$

Letting $n = 1$, $m = 2$, we have

$$d(T\xi, T^2\xi) = d(T^{k+1}\xi, T^{k+2}\xi) \ , \qquad k = 1,2,\ldots \ .$$

We infer that $d(T\xi, T^2\xi) = 0$, since T satisfies Bailey's condition. That is, $T\xi$ is a fixed point under T. It is also clear that $T\xi \in \ell(x)$ for T is continuous and $\xi \in \ell(x)$. Therefore, Theorem 1 implies $\ell(x) = \{T\xi\}$, so that $\xi = T\xi$.

Remark:

Bailey proves a similar result in [1] using his condition on T and also requiring that T be continuous on a compact metric space X.

Corollary (Edelstein [5])

Let $T:X \to X$ be a mapping such that for $x \neq y$, $d(Tx,Ty) < d(x,y)$, where $x,y \in X$. Suppose there exists $x \in X$ such that the sequence $\{T^n x\}_{n=o}^{\infty}$ has a convergent subsequence whose limit is ξ. Then ξ is a unique fixed point of T.

Example I.1:

This example will show Theorem I.2 is a non-vacuous generalization of Edelstein's result.

Let X be the subset of the plane defined by

$$X = \left\{\left(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}\right) : n = 0,1,2,\ldots\right\} \cup \left\{\left(\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^n-1}{2^n}\right) : n = 0.1.2,\ldots\right\} \cup \{(1,1)\} \ .$$

Define the mapping $T:X \to X$ in the following way:

$$\left(\frac{2^n-1}{2^n}, \frac{2^n-1}{2^n}\right) \to \left(\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^n-1}{2^n}\right) \to \left(\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^{n+1}-1}{2^{n+1}}\right) .$$

and $(1,1) \to (1,1)$.

It is obvious that T can not satisfy the assumption in the Corollary to Theorem I.2. However, it is easy to see that T satisfies all hypothesis in Theorem I.2 and hence has a unique fixed point, namely, (1,1).

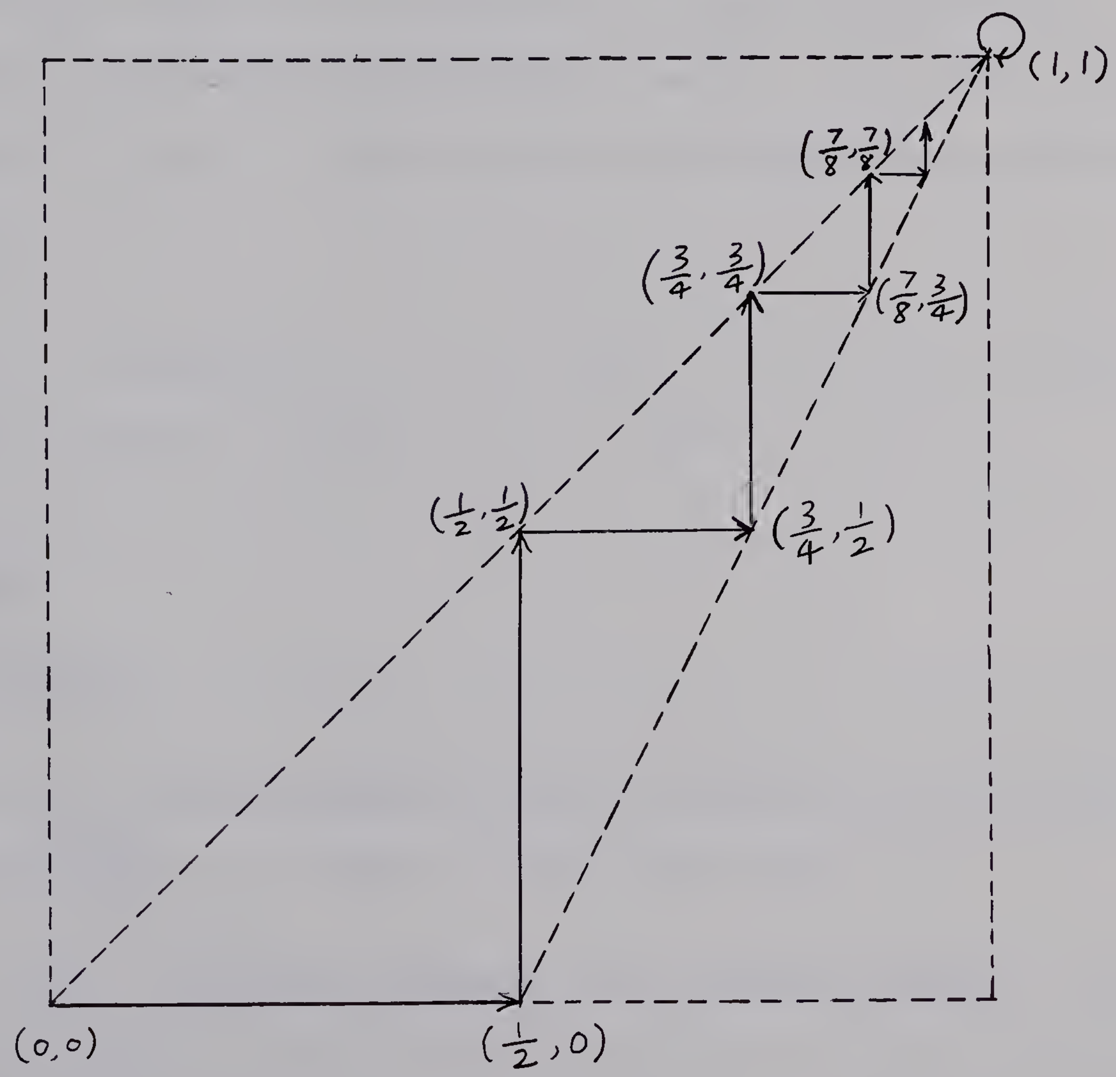
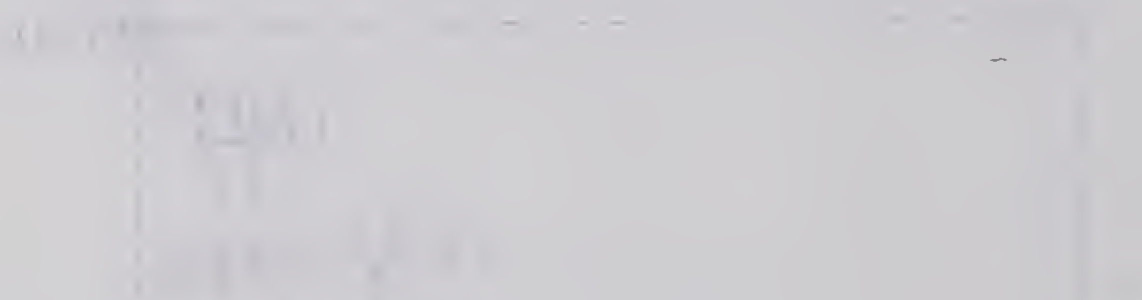


Figure for Example I.1



Remark:

In this example, X can be modified in an obvious manner so that it is not compact. Hence the hypothesis for Bailey's theorem in [1] is not satisfied either.

CHAPTER II

FIXED POINTS OF SEQUENCE OF MAPPINGS

Let (X,d) be a metric space and $\{T_n\}_{n=1}^{\infty}$ a sequence of mapping from X into itself converging to a mapping $T:X \to X$. Let a_n and a be respectively fixed points of T_n and T. We are concerned with the following problem: When will the convergence of $\{T_n\}_{n=1}^{\infty}$ to T imply the convergence of $\{a_n\}_{n=1}^{\infty}$ to a? Some conditions under which this problem can be answered are given along with some applications to differential and integral equations. The results of F. F. Bonsall [2] and S. B. Nadler [10] are included as easy corollaries of our work.

1° A mapping $T:X \quad X$ is a Banach contraction if and only if $d(Tx,Ty) \leq \alpha d(x,y)$ for all $x,y \in X$, where $0 \leq \alpha < 1$.

<u>Theorem II.1</u>.

Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is a sequence of Banach contractions with the same contractive constant α. Each T_n has a fixed point a_n.

(ii) $\{T_n\}_{n=1}^{\infty}$ converges pointwise to an arbitrary mapping $T:X \to X$ with a fixed point a.

Then the sequence $\{a_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point

a ; consequently T has only one fixed point.

Proof:

Let α be the common contractive constant of $\{T_n\}_{n=1}^{\infty}$, then $0 \leq \alpha < 1$ and $d(T_n x, T_n y) \leq \alpha d(x,y)$ for any n and $x,y \in X$.

Since $\{T_n\}_{n=1}$ converges pointwise to T , given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(Ta, T_n a) < (1-\alpha)\varepsilon \quad .$$

Thus for $n \geq N$,

$$\begin{aligned} d(a,a_n) &= d(Ta, T_n a_n) \\ &\leq d(Ta, T_n a) + d(T_n a, T_n a_n) \\ &< (1-\alpha)\varepsilon + \alpha d(a,a_n) \ , \end{aligned}$$

i.e. $(1-\alpha)d(a,a_n) < (1-\alpha)\varepsilon$.

Since $0 \leq \alpha < 1$, we have $d(a,a_n) < \varepsilon$ for $n \geq N$, so that $\lim_{n \to \infty} a_n = a$.

Suppose a' is another fixed point of T , then by above argument, $a' = \lim_{n \to \infty} a_n$. Hence T can have only one fixed point.

Corollary (Bonsall [2])

Let $\{T_n\}_{n=1}^{\infty}$ be a convergent sequence of Banach contractions with the same contraction constant from a complete metric space X into itself. Suppose the limit mapping $Tx = \lim_{n \to \infty} T_n x$ is also a Banach

contraction with the same contraction constant. Then the sequence $\{a_n\}_{n=1}^{\infty}$ of fixed points of $\{T_n\}_{n=1}^{\infty}$ converges to the fixed point a of T.

Example II.1

Let $\{f_n(t,x)\}_{n=1}^{\infty}$ be a uniformly bounded sequence of continuous, real-valued functions defined in an open connected subset D of the plane, which converges pointwise to a continuous function $f(t,x)$ defined in D. Suppose there is some constant $K > 0$ such that

$$|f_n(t,x_1) - f_n(t,x_2)| \leq K|x_1-x_2|, \quad \text{for} \quad (t,x_1),(t,x_2) \in D \quad .$$

Let $\phi_n(t)$ be the solution on $I = [a,b]$, $(K(b-a) < 1)$ of the initial value problem

$$x'(t) = f_n(t,x) \ , \qquad x(t_n) = \xi_n \quad ,$$

where $(t_n,\xi_n) \in D$ and $\lim_{n \to \infty} (t_n,\xi_n) = (t_o,\xi_o)$. (We understand that the curves $\phi_n(t)$, $a \leq t \leq b$, $n = 1,2,\ldots$ lie in D.) Suppose also $\phi(t)$ is a solution on I of the initial value problem

$$x'(t) = f(t,x) \ , \qquad x(t_o) = \xi_o \quad .$$

Then $\{\phi_n(t)\}_{n=1}^{\infty}$ converges uniformly on I to $\phi(t)$ and furthermore $\phi(t)$ is the unique solution on I.

Remark:

J. R. Dorroh in [10] assumed the additional hypothesis that the limit function $f(t,x)$ is also Lipschitzian with constant K, but

the above example shows that all that is needed is the continuity of f.

Proof of Example II.1.

Let $C(I)$ be the space of all real continuous functions defined on I with the metric of uniform convergence. Let T_n, $T: C(I) \to C(I)$ be defined by

$$(T_n x)t = \xi_n + \int_{t_n}^{t} f_n(s,x(s))ds\ , \qquad t \in I\ , \quad x \in C(I)\ .$$

$$(Tx)t = \xi_o + \int_{t_o}^{t} f(s,x(s))ds\ , \qquad t \in I\ , \quad x \in C(I)\ .$$

Since $f_n(t,x)$ satisfies a Lipschitz condition and $K(b-a) < 1$, $\{T_n\}_{n=1}^{\infty}$ is a sequence of Banch contractions on $C(I)$ with the same contractive constant $K(b-a)$. To apply Theorem II.1, we need only prove $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T on $C(I)$.

Let $x \in C(I)$ and $y = Tx$, $y_n = T_n x$, then

$$|y(t)-y_n(t)| \le |\xi_o-\xi_n| + \left|\int_{t_o}^{t} \{f(s,x(s))-f_n(s,x(s))\}ds\right|$$
$$+ \left|\int_{t_n}^{t_o} f_n(s,x(s))ds\right|\ .$$

Since $\{f_n\}$ are uniformly bounded, the last term on the right hand side tends to zero. Also, by the Lebesque bounded convergence theorem the second term on the right hand side tends to zero. Therefore $\{y_n(t)\}$ converges pointwise on $[a,b]$. The uniform convergence of $\{y_n(t)\}$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

to $y(t)$ follows from the equicontinuity of $\{y_n(t)\}$ on the compact interval $[a,b]$. Hence $\{T_n\}$ converges pointwise in $C(I)$ to T.

Example II.2.

Let $K(x,y)$ and $K_n(x,y)$, $n = 1,2,\ldots$, be continuous real valued functions defined in $[a,b] \times [a,b]$ such that $K(x,y) = \lim_{n \to \infty} K_n(x,y)$. Suppose

$$|K_n(x_1,y) - K_n(x_2,y)| \leq L|x_1-x_2| \quad .$$

and $|K_n(x,y)| \leq M$ for $a \leq y \leq b$, where $M(b-a) < 1$.

Let $\phi_n(x)$ be a solution of the integral equation

$$f(x) = \int_a^b K_n(x,y)f(y)dy + g_n(x) \quad , \qquad x \in C[a,b] \quad ,$$

where $\{g_n(x)\}$ is a sequence of continuous functions converging uniformly to $g(x)$ on $[a,b]$.

Suppose the integral equation

$$f(x) = \int_a^b K(x,y)f(y)dy + g(x) \quad ,$$

has a solution $\phi(x)$. Then this solution is unique and $\lim_{n \to \infty} \phi_n(x) = \phi(x)$ uniformly on $[a,b]$.

Proof.

Consider the mappings T, T_n in $C[a,b]$:

$$(Tf)x = \int_a^b K_n(x,y)f(y)dy + g(x) \quad , \quad x \in [a,b]$$

$$(T_n f)x = \int_a^b K(x,y)f(y)dy + g_n(x) \quad , \quad x \in [a,b] \quad .$$

It can be verified that $\{T_n\}_{n=1}^{\infty}$ is a sequence of Banach contractions on $C[a,b]$ with the same constant $M(b-a) < 1$. By using an argument similar to the proof of Example II.1 we can show that $\{T_n\}_{n=1}^{\infty}$ converges pointwise to T. Hence by Theorem II.1, $\{\phi_n(x)\}$ converges uniformly to $\phi(x)$ and $\phi(x)$ is the unique solution.

2^{o} In some cases of interest, the sequence $\{T_n\}_{n=1}^{\infty}$ of Banach contractions may not have the same contractive constant; <u>even more generally</u>, the mappings $\{T_n\}$ may not be Banach contactions. In this regard, we have the following generalization of Nadler's theorem [10]:

<u>Theorem II.2</u>.

Suppose

(i) $T:X \to X$ is a Banach contraction with a fixed point a.

(ii) $T_n:X \to X$ has a fixed point a_n, $n = 1,2,\ldots$.

(iii) $\{T_n\}$ converges uniformly to T on the subset $\{a_n : n = 1,2,\ldots\}$.

Then the sequence $\{a_n\}$ converges to a.

<u>Proof</u>:

Let α be the contractive constant of T, $0 \le \alpha < 1$,

$$d(Tx,Ty) \le \alpha d(x,y) \quad \text{for any} \quad x \neq y \quad .$$

By uniform convergence on the subset $\{a_n : n = 1,2,\ldots\}$, given $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(T_n a_n, Ta_n) < (1-\alpha)\varepsilon \quad .$$

Hence for $n \geq N$,

$$\begin{aligned} d(a,a_n) &= d(Ta,T_n a_n) \\ &\leq d(Ta,Ta_n) + d(Ta_n,T_n a_n) \\ &< \alpha d(a,a_n) + (1-\alpha)\varepsilon \quad , \end{aligned}$$

i.e. $\quad (1-\alpha)d(a,a_n) < (1-\alpha)\varepsilon \quad , \qquad 0 \leq \alpha < 1 \quad ,$

so that $d(a,a_n) < \varepsilon$.

Remark:

(1) The result of S. B. Nadler [10] is a corollary of above theorem. He assumes the space X to be complete and each T_n is a Banach contraction to yield the same conclusion.

(2) If in Theorem 1, we assume $\{T_n\}_{n=1}^{\infty}$ to be an equicontinuous sequence of mappings such that $T_n(X)$ is a compact subset of X , then pointwise convergence instead of uniform convergence is sufficient.

Example II.3.

Let $\{f_n(t,x)\}_{n=1}^{\infty}$ be a sequence of continuous real functions defined in a compact domain of the plane and $\phi_n(t)$ a solution on $I = [a,b]$ of the initial value problem: $x'(t) = f_n(t,x)$, $x(t_n) = \xi_n$. (It is understood that $(t,\phi_n(t)) \in D$ and $t \in I$.) Suppose

$\lim_{n \to \infty} (t_n,\xi_n) = (t_o,\xi_o) \in D$ and $\{f_n(t,x)\}$ converges uniformly on D to a function $f(t,x)$ which satisfies the Lipschitz condition in D :

$$|f(t,x_1)-f(t,x_2)| \leq L|x_1-x_2| \quad \text{for} \quad (t,x_1),(t,x_2) \in D \quad ,$$

where $L(b,a) < 1$. Then the sequence $\{\phi_n(t)\}$ of solutions converges uniformly on I to the solution $\phi(t)$ on I of the initial value problem

$$x'(t) = f(t,x) \ , \qquad x(t_o) = \xi_o \quad .$$

Remark:

The restriction that the function is Lipschitzian can be removed. We will do this in Example II.5. See also Hartman's book [7].

Proof of Example II.3.

Let $T, T_n : C[a,b] \to C[a,b]$ defined as

$$(Tx)t = \xi_o + \int_{t_o}^{t} f(s,x(s))ds \ , \qquad t \in [a,b]$$

$$(T_n x)t = \xi_n + \int_{t_o}^{t} f_n(s,x(s))ds \ , \qquad t \in [a,b] \quad .$$

It is easy to verify that T is a Banach contraction. Since $(t_n,\xi_n) \to (t_o,\xi_o)$ and $f_n(t,x) \to f(t,x)$ uniformly on D , the uniform convergence of $\{T_n\}$ to T follows from the following estimate:

$$\|Tx-T_nx\| = \sup_{a \le t \le b} |Tx(t) - T_nx(t)|$$

$$\le |\xi_o-\xi_n| + \int_a^b |f(s,x(s))-f_n(s,x(s))|ds + \int_{t_n}^{t_o} |f_n(s,x(s))|ds$$

$$\le |\xi_o-\xi_n| + (b-a) \sup_{(t,x)\in D} |f(t,x)-f_n(t,x)| + M|t_o-t_n| ,$$

where M is the uniform bound of $\{f_n\}$. (Such M exists since the convergence is uniform and each f_n is continuous.) The conclusion follows by Theorem II.2.

Example II.4.

Let $\phi_n(t)$ be a solution of the non-linear integral equation

$$f(x) = \int_a^b K_n(x,y,f(y))dy + g_n(x) , \quad x \in [a,b] ,$$

where $K_n(x,y,z)$ and $g_n(x)$ are continuous $((x,y,z) \in [a,b] \times [a,b] \times [a,b])$.

Suppose K_n and g_n converge uniformly to $K(x,y,z)$ and $g(x)$ respectively, where $K(x,y,z)$ satisfies the Lipschitz condition

$$|K(x,y,z_1) - K(x,y,z_2)| \le L|z_1-z_2| , \quad L(b-a) < 1 .$$

Let $\phi(x)$ be the solution of the integral equation

$$f(x) = \int_a^b K(x,y,f(y))dy + g(x) , \quad x \in [a,b] .$$

Then $\{\phi_n(x)\}_{n=1}^{\infty}$ converges uniformly to $\phi(x)$ on $[a,b]$.

Proof.

The mapping $T\colon C[a,b] \to C[a,b]$ given by

$$Tf(x) = \int_a^b K(x,y,f(y))dy + g(x)$$

is a Banach contraction mapping. It can be shown that the sequence of mappings T_n defined by

$$T_n f(x) = \int_a^b K_n(x,y,f(y))dy + g_n(x)$$

converges uniformly to T on $C[a,b]$.

Remark:

Theorem II.2 can be further extended. A mapping $T:X \to X$ is said to satisfy Meir's condition [9] if for any $\varepsilon > 0$ there exits $\lambda(\varepsilon) > 0$ such that $d(x,y) > \varepsilon$ implies $d(Tx,Ty) < d(x,y)-\lambda(\varepsilon)$. Any Banach contraction satisfies Meir's condition. Indeed, given $\varepsilon > 0$, let $\lambda(\varepsilon) = (1-\alpha)\varepsilon$, then $d(x,y) > \varepsilon$ implies

$$\begin{aligned} d(Tx,Ty) \leq \alpha d(x,y) &= d(x,y) - (1-\alpha)d(x,y) \\ &< d(x,y) - (1-\alpha)\varepsilon \\ &= d(x,y) - \lambda(\varepsilon) \quad . \end{aligned}$$

Theorem II.3.

Suppose

(i) $T:X \to X$ satisfies Meir's condition and $Ta = a$.

(ii) $T_n : X \to X$ has a fixed point a_n, $n = 1,2,\ldots$.

(iii) $\{T_n\}$ converges uniformly to T on the subset $\{a_n : n = 1,2,\ldots\}$.

Then the sequence $\{a_n\}$ converges to a .

Proof.

Suppose $\{a_n\}$ does not converge to a , then there exists $\varepsilon > 0$ and a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $d(a,a_{n_k}) > \varepsilon$. By Meir's condition there exists $\lambda(\varepsilon) > 0$ such that

$$d(Ta,Ta_{n_k}) < d(a,a_{n_k}) - \lambda(\varepsilon) \quad ,$$

so that $d(a,a_{n_k}) - d(Ta,Ta_{n_k}) > \lambda(\varepsilon)$.

By the triangle inequality,

$$\begin{aligned} d(T_{n_k} a_{n_k}, Ta_{n_k}) &\geq d(T_{n_k} a_{n_k}, Ta) - d(Ta,Ta_{n_k}) \\ &= d(a_{n_k}, a) - d(Ta,Ta_{n_k}) \\ &> \lambda(\varepsilon) > 0 \quad , \end{aligned}$$

contradicting the uniform convergence of T_n on the subset $\{a_n : n = 1,2,\ldots\}$.

3° In the theorems of section 2°, we assume the convergence is uniform. If we assume some additional conditions on the space X and the mappings, the uniformity of convergence can be removed.

A mapping $T:X \to X$ is called a Bailey contraction [1] if for any pair of distinct points $x,y \in X$, there is a positive integer

$n = n(x,y)$ depending on x and y such that $d(T^n x, T^n y) < d(x,y)$, where T^n is defined as $T^o x = x$, $T^n x = T(T^{n-1} x)$, $n = 1,2,\ldots$, D. F. Bailey [1] proved that if X is compact, a continuous Bailey contraction has a unique fixed point. However, it is not known whether this result can be extended to locally compact spaces.

Theorem II.4.

Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of Bailey contractions on a locally compact metric space X.

(ii) $\{T_n\}$ converges pointwise to a Banach contraction T with fixed point a.

Then for sufficiently large n, each T_n has a unique fixed point a_n; furthermore the sequence $\{a_n\}$ converges to a.

Proof.

Since X is locally compact, the fixed point a has a compact neighbourhood K, and hence T_n converges uniformly on K.

Let S be a closed sphere centered at a with radius r, contained in the compact subset K, then S is also compact. We show that there exists a positive integer N such that for $n \geq N$ $T_n(S) \subset S$. Indeed, we can choose an N by means of uniform convergence such that $n \geq N$ implies

$$d(Tx, T_n x) < (1-\alpha)r \qquad \text{for all} \quad x \in S ,$$

where α is the contractive constant of T ; consequently for $x \varepsilon S$ and $n \geq N$ we have

$$\begin{aligned} d(a,T_n x) &= d(Ta,T_n x) \\ &\leq d(Ta,Tx) + d(Tx,T_n x) \\ &< \alpha d(a,x) + (1-\alpha)r \\ &\leq \alpha r + (1-\alpha)r = r \quad . \end{aligned}$$

Now the restiction of T_n $(n \geq N)$ to S is a continuous Bailey contraction of the compact space S so by a theorem of Bailey [1], T_n $(n \geq N)$ has a fixed point a_n in S .

Furthermore, since T is a Banach contraction and $\{T_n\}$ converges uniformly to T on S . Theroem II.2 implies $\lim\limits_{n \to \infty} a_n = a$. This completes the proof.

Corollary.

Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is a sequence of mappings from a locally compact space X into itself and satisfies the following condition:

$$d(T_n x,T_n y) < d(x,y) \qquad \text{for} \quad x \neq y \ , \quad x,y \ \varepsilon \ X \quad .$$

(ii) $\{T_n\}$ converges pointwise to a Banach contraction T with fixed point a .

Then for sufficiently large n , T_n has a unique fixed point a_n ; furthermore the sequence a_n converges to a .

Proof.

It is clear that the sequence $\{T_n\}$ is equicontinuous and that the T_n's are Bailey contractions.

Remark.

S. B. Nadler [10] proved a special case of Theorem II.2 by assuming each T_n to be a Banach contraction and the space to be complete.

A result similar to Theorem II.4 can be obtained for a sequence of mappings having the property of diminishing obital diameters [8], which will be defined below. Let $T:X \to X$, and define the orbit $O(x)$ of a point $x \in X$ to be the set $\{T^m x : m = 0,1,2,\ldots\}$. Denote by $\delta(A)$ the diameter of the subset $A \subset X$. We see that $\{\delta(O(T^n x))\}_{n=o}^{\infty}$ is a non-increasing sequence of non-negative numbers and hence has a limit $r(x)$. Following W. A. Kirk [8], we say that T has a diminishing orbital diameter if $\delta(O(x)) > r(x) = \lim_{n \to \infty} \delta(O(T^n x))$. In case X is compact, W. A. Kirk [8] proved every continuous mapping having diminishing orbital diameter has at least one fixed point.

Theorem II.5.

Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of mappings having diminishing orbital diameter on a locally compact space X.

(ii) $\{T_n\}$ converges pointwise to a Banach contraction T with fixed point a.

Then for sufficiently large n, each T_n has a fixed point a_n;

furthermore, the sequence $\{a_n\}$ converges to a .

Proof.

The argument is similar to the proof of Theorem II.4.

4^o We now consider the converse problem: suppose we do not know about the existence of fixed points of the limit mapping T and suppose each T_n has a fixed point a_n . Can we conclude the existence of any fixed point of T from subsequential convergence of $\{a_n\}$? The following theorem gives a partial answer to this question.

Theorem II.6.

Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of mappings from X into X , each of which has a fixed point a_n .

(ii) $\{T_n\}$ converges pointwise to any mapping $T:X \to X$.

(iii) $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ whose limit is a .

Then a is a fixed point of T .

Proof.

Since the sequence $\{T_n\}$ is equicontinuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(T_n x, T_n y) < \varepsilon/2$, for all n . On the other hand for $\delta > 0$ there exists $N(\delta)$ such that $k \geq N$ implies $d(a,a_{n_k}) < \delta$. Hence for $k \geq N(\delta)$, we have $d(T_{n_k} a, T_{n_k} a_{n_k}) < \varepsilon/2$. Therefore for sufficiently large k ,

$$d(Ta,a_{n_k}) = d(Ta,T_{n_k}a_{n_k})$$

$$\leq d(Ta,T_{n_k}a) + d(T_{n_k}a,T_{n_k}a_{n_k})$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

We have proved $Ta = \lim_{k\to\infty} a_{n_k}$, so $Ta = a$.

<u>Theorem II.7</u>.

Suppose

(i) $\{T_n\}$ is any sequence of mappings from X into X with fixed points $\{a_n\}$, converging uniformly to a continuous mapping T .

(ii) $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ whose limit is a .

Then a is a fixed point of T .

<u>Proof.</u>

The inequality,

$$d(Ta,a_{n_k}) = d(Ta,T_{n_k}a_{n_k})$$

$$\leq d(Ta,Ta_{n_k}) + d(Ta_{n_k},T_{n_k}a_{n_k}) ,$$

implies $a_{n_k} \to Ta$, since T is continuous and the sequence $\{T_n\}$ converges to T uniformly.

<u>Example II.5</u>. (See Theorem 2.4 of [7])

We will extend the result obtained in Example II.3.

Suppose $\{f_n(t,x)\}$ is a sequence of continuous functions defined on a compact domain D in the plane, converging uniformly to a continuous function $f(t,x)$ on D and $\{(t_n,\xi_n)\}$ is a convergent sequence of points in D with limit $(t_o,\xi_o) \varepsilon D$. Suppose $\phi_n(t)$ is a solution on $[a,b]$ of the initial value problem: $x'(t) = f_n(t,x), x'(t_n) = \xi_n$. Then $\{\phi_n\}$ has a uniformly convergent subsequence $\{\phi_{n_k}\}$ whose limit is a solution on $[a,b]$ of

$$x'(t) = f(t,x) , \quad x(t_o) = \xi_o \quad .$$

Moreover, if $\phi(t)$ is the unique solution, then $\{\phi_n\}$ converges uniformly to $\phi(t)$.

<u>Proof</u>.

Let T , $T_n : C[a,b] \to C[a,b]$ be defined as in Example II.3,

$$(T_n x)t = \xi_n + \int_{t_n}^{t} f_n(s,x(s))ds$$

$$(Tx)t = \xi_o + \int_{t_o}^{t} f(s,x(s))ds \quad .$$

T is continuous since $f(t,x)$ is continuous on D .

Since

$$\phi_n(t) = \xi_n + \int_{t_n}^{t} f_n(s,\phi_n(s))ds \ ,$$

$\{\phi_n\}$ is equicontinuous and uniformly bounded and hence has a uniformly convergent subsequence $\{\phi_{n_k}\}$ with limit $\phi(t)$, say. Now Theorem II.7

implies $\phi(t)$ is a solution on $[a,b]$ of

$$x'(t) = f(t,x) , \quad x(t_o) = \xi_o .$$

Finally, if $\phi(t)$ is the unique solution then every uniformly convergent subsequence of $\{\phi_n\}$ has the same limit $\phi(t)$. Since $\{\phi_n\}$ is equicontinuous and uniformly bounded on $[a,b]$, $\{\phi_n\}$ converges uniformly to $\phi(t)$. Indeed, if there is an $\varepsilon > 0$ and a subsequence $\{\phi_{n_k}(t)\}$ such that $|\phi_{n_k}(t)-\phi(t)| > \varepsilon$ for all $t \varepsilon [a,b]$ and any k, then $\{\phi_{n_k}(t)\}$ has no uniformly convergent subsequence. This leads to a contradiction since $\{\phi_{n_k}(t)\}$ is equicontinuous and uniformly bounded.

CHAPTER III

PERIODICITY AND A CLASS OF MAPPINGS

Much current research is concerned with the fixed points of contractive mappings (mappings which shrink distance in some manner) from a metric space into itself. In this Chapter we shall point out that most mappings treated in the literature are very special in the sense that all these mappings satisfy a condition which is rather severe: every periodic point must necessarily be a fixed point.

We list some of these contractive conditions below:

(1) (Banach): There is a number α , $0 \leq \alpha < 1$ such that $d(Tx,Ty) \leq \alpha d(x,y)$, $x,y \in X$.

(2) (Rakotch [11]): There exists a decreasing function $\alpha(d(x,y))$ depending on the metric $d(x,y)$, $0 \leq \alpha(d(x,y)) < 1$; such that $d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$. $x,y \in X$.

(3) (Boyd and Wong [12]): for $x \neq y$, $d(Tx,Ty) \leq \psi(d(x,y))$, where $\psi(d)$ is an upper semi-continuous function of the metric d and $\psi(d) < d$ for $d > 0$; furthermore $\liminf_{d \to \infty} \{d - \psi(d)\} > 0$.

(4) (Meir [9]): Given $\varepsilon > 0$, there exists $\lambda(\varepsilon) > 0$ such that $d(x,y) > \varepsilon$ implies $d(Tx,Ty) < d(x,y) - \lambda(\varepsilon)$.

(5) (Edelstein [5]): $d(Tx,Ty) < d(x,y)$ for all $x \neq y$.

(6) (Bailey [1]): for all $x \neq y$, there exists $n = n(x,y)$ such that $d(T^n x, T^n y) < d(x,y)$.

(7) (Belluce and Kirk [8]): If $\delta(O(x)) > 0$ then $\lim_{n \to \infty} \delta(O(T^n x)) < \delta(O(x))$, where the notations are refered to Theorem II.5.

It is obvious that a mapping satisfying any one of (1), (2), (3), and (4) will satisfy (5) and in turn, condition (5) implies condition (6).

A mapping $T:X \to X$ is called NON-PERIODIC if $x \neq Tx$ implies $x \neq T^n x$ for all $n = 1,2,\ldots$.

We now show that a mapping satisfying any one of the conditions (1) to (7) is a non-periodic mapping. It is sufficient to show this for mappings satisfying condition (6) and (7).

Theorem III.1

A mapping $T:X \to X$ is non-periodic if it satisfies condition (6): For $x \neq y$ there exist $n = n(x,y)$ such that

$$d(T^n x, T^n y) < d(x,y) \quad .$$

Proof

Suppose $x \neq Tx$ and there exists some positive integer K which is the smallest such that $T^K x = x$.

By hypothesis we can choose $n_1(x)$ which is the least positive

integer such that

$$d(x,Tx) > d(T^{n_1}x, T^{n_1+1}x) \quad .$$

Observe that $n_1 < K$ and $d(T^{n_1}x, T^{n_1+1}x) > 0$. Indeed if $n_1 \geq K$, then $n_1 = rK+q$ where r,q are positive integers , $0 \leq q < K \leq n_1$; consequently

$$d(x,Tx) > d(T^{n_1}x, T^{n_1+1}x) = d(T^q x, T^{q+1}x) \quad ,$$

Contradicting minimality of n_1 . Also, if $d(T^{n_1}x, T^{n_1+1}x) = 0$, then $T^{n_1}x = T^{n_1+1}x$, hence $T^{n_1+(k-n_1)}x = T^{n_1+1+(k-n_1)}x$, i.e. $x = Tx$, contradicting our assumption on x .

Now since $d(T^{n_1}x, T^{n_1+1}x) > 0$, we can select $n_2(x)$ as the smallest positive integer such that

$$d(T^{n_1}x, T^{n_1+1}x) > d(T^{n_2}x, T^{n_2+1}x) \quad .$$

The same argument as above is used to deduce that $n_2 < K$ and $d(T^{n_2}x, T^{n_2+1}x) > 0$.

Proceeding in this manner, we can find a sequence $\{n_i\}$ of positive integers such that $n_i < K$ and

$$d(x,y) > d(T^{n_1}x, T^{n_1+1}x) > d(T^{n_2}x, T^{n_2+1}x) > \ldots \quad .$$

But then there must be two indices, say $i > j$ such that $n_i = n_j$ since $n_i < K$, $i = 1,2,\ldots$. This is a contradiction, for then

$$d(T^{n_j}x, T^{n_j+1}x) = d(T^{n_i}x, T^{n_i+1}x) \quad .$$

Theorem III.2

A mapping $T:X \to X$ is non-periodic if it satisfies condition (7): If $\delta(O(x)) > 0$ then

$$\lim_{n \to \infty} \delta(O(T^n x)) < \delta(O(x)) \quad .$$

Proof

We first note that $\delta(O(x)) > 0$ if and only if $x \neq Tx$. Also, by definition of $O(x)$,

$$\delta(O(x)) \geq \delta(O(Tx)) \geq \ldots \geq \delta(O(T^n x)) \geq \ldots \geq \lim_{n \to \infty} \delta(O(T^n x)) \quad .$$

Suppose $x \neq Tx$, then by hypothesis we have

$$\delta(O(x)) > \lim_{n \to \infty} \delta(O(T^n x)) \quad .$$

Hence there is an N such that $\delta(O(x)) > \delta(O(T^N x))$, so we have $O(x) \supsetneq O(T^N x)$. This implies that $x \notin O(T^N x)$, i.e. $x \neq T^N x$, $T^{N+1}x, \ldots$.

In addition, it is impossible that $x = T^m x$ for $m < N$. For if so, let $p > 0$ be an integer such that m divides $N+p$ (such integer exists by the Euclidean Algorithm), then $x = T^{N+p}x$, contradicting the argument in the previous paragraph.

BIBLIOGRAPHY

1. D.F. Bailey, Some theorems on contractive mappings, J. London Math. Soc., 41(1966), 101-106.

2. F.F. Bonsall, Lectures on some fixed points theorems of functional analysis, Tata Institue of Fundamental Research, Bombay, India, 1962.

3. J. Cano, On communtating mappings by successive approximations, Amer. Math. Monthly 75(1968), 393-394.

4. S.C. Chu and J.B. Diaz, Remarks on a generalization of Banach's principle of contractive mappings, J. of Math. Analysis and Applications, Vol. II, No. 1-3 (1965).

5. M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37(1962), 74-79.

6. M. Edelstein, On non-expansive mappings, Proc. Amer. Math. Soc. 15(1964), 689-

7. P. Hartman, Ordinary differential equations, John Wiley and Sons, 1964.

8. W. A. Kirk, On Mappings with Diminishing Orbital Diameters. (To appear).

9. A. Meir, A theorem on contractive mappings, (To appear).

10. S.B. Nadler, Jr., Sequence of Contractions and Fixed Points. (To appear).

11. E. Rakotch, A note on contractive mappings. Proc. Amer. Math. Soc. 13(1962), 459-465.

12. D.W. Boyd and J.S.W. Wong, On non-linear contractions. (To appear).